

# Generalized Cohn's Theorem

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## Abstract

We introduce the notion of a free associative  $\mathcal{Z}_2$ -algebra on the union of two disjoint sets and prove a generalization of Cohn's Theorem on Jordan algebras.

It is well-known that the ordinary passage from an associative algebra  $\mathcal{A}$  to a Lie algebra is established by the following binary operation:

$$[x, y] = xy - yx \quad \text{for } x, y \in \mathcal{A}. \quad (1)$$

Replacing the sign  $-$  by the sign  $+$  and inserting a scalar factor  $\frac{1}{2}$  on the right hand side of (1), the resulting binary operation:

$$x \circ y = \frac{1}{2}(xy + yx) \quad \text{for } x, y \in \mathcal{A} \quad (2)$$

gives rise to the passage from an associative algebra to a Jordan algebra.

As a generalization of Lie algebras, (right) Leibniz algebras have been studied recently by a group of researchers. If  $A$  is an associative  $\mathcal{Z}_2$ -algebra, then the binary operation:

$$\langle x, y \rangle = xy_0 - y_0x \quad \text{for } x, y \in A \quad (3)$$

gives a passage from an associative  $\mathcal{Z}_2$ -algebra to a (right) Leibniz algebra, where  $y_0$  is the even component of  $y$ . Replacing the sign  $-$  by the sign  $+$  and inserting a scalar factor  $\frac{1}{2}$  on the right hand side of (3), the resulting binary operation:

$$x \bullet y = \frac{1}{2}(xy_0 + y_0x) \quad \text{for } x, y \in A \quad (4)$$

introduces a class of non-associative algebras  $(A, \bullet)$ . Since the way of producing the binary operation (4) from the binary operation (3) is the same as the way of producing the binary operation (2) from the binary operation (1), we name the

class of non-associative algebras  $(A, \bullet)$  the generalized Jordan algebras in [2]. Thus, the connection between Leibniz algebras and generalized Jordan algebra extends the connection between Lie algebras and Jordan algebra. Hence, if Leibniz algebras should be explored, then the generalized Jordan algebras should not be ignored.

The purpose of this paper is to prove a generalization of Cohn's Theorem on Jordan algebras in the context of the generalized Jordan algebras. In Section 1, we discuss the basic properties of a generalized Jordan algebra and establish the passage from an associative  $\mathcal{Z}_2$ -algebra to a generalized Jordan algebra. In Section 2, we state the universal property of the free associative  $\mathcal{Z}_2$ -algebra on the union of two disjoint sets. In Section 3, we introduce odd tetrads and extend Cohn's Theorem in the context of the generalized Jordan algebras.

In this paper, all vector spaces are vector spaces over fields of characteristic not 2 and 3, and all associative algebras have an identity.

## 1 Generalized Jordan Algebras

We begin this section with the definition of a generalized Jordan algebra, which was introduced in [2].

**Definition 1.1** *A vector space  $J$  is called a **generalized Jordan algebra** if there exists a binary operation  $\bullet : J \times J \rightarrow J$  such that the following two properties hold.*

(i) *The binary operation  $\bullet$  is **right commutative**; that is*

$$x \bullet (y \bullet z) = x \bullet (z \bullet y) \quad \text{for } x, y, z \in J. \quad (5)$$

(ii) *The binary operation  $\bullet$  satisfies the **Jordan identity**:*

$$(y \bullet x) \bullet (x \bullet x) = (y \bullet (x \bullet x)) \bullet x \quad (6)$$

*and the **Hu-Liu identity**:*

$$x \bullet (y \bullet (x \bullet x)) - (x \bullet y) \bullet (x \bullet x) = 2(x \bullet x) \bullet (y \bullet x) - 2((x \bullet x) \bullet y) \bullet x, \quad (7)$$

*where  $x, y \in J$ .*

A generalized Jordan algebra  $J$  is also denoted by  $(J, +, \bullet)$ , where the binary operation  $\bullet$  is called the **bullet product**. If there exists an element 1 of a generalized Jordan algebra  $(J, +, \bullet)$  such that  $x \bullet 1 = x$  for  $x \in J$ , then  $J$  is said to be **right unital** and the element 1 is called a **right unit**.

**Definition 1.2** *An associative algebra  $A$  is called an **associative  $\mathcal{Z}_2$ -algebra** if  $A = A_0 \oplus A_1$  (as vector spaces) and*

$$A_0 A_0 \subseteq A_0, \quad A_0 A_1 + A_1 A_0 \subseteq A_1 \quad \text{and} \quad A_1 A_1 = 0. \quad (8)$$

If  $A = A_0 \oplus A_1$  is an associative  $\mathcal{Z}_2$ -algebra, then  $A_0$  and  $A_1$  are called the **even part** and **odd part** of  $A$  respectively. An element  $x$  of  $A = A_0 \oplus A_1$  can be written uniquely as  $x = x_0 + x_1$ , where  $x_0 \in A_0$  and  $x_1 \in A_1$  are called the **even component** and **odd component** of  $x$ , respectively.

The following proposition establishes the passage from an associative  $\mathcal{Z}_2$ -algebra to a right unital generalized Jordan algebra.

**Proposition 1.1** *If  $A = A_0 \oplus A_1$  is an associative  $\mathcal{Z}_2$ -algebra, then  $A$  becomes a right unital generalized Jordan algebra under the following bullet product*

$$x \bullet y = \frac{1}{2}(xy_0 + y_0x), \quad (9)$$

where  $x, y \in A$  and  $y_0$  is the even component of  $y$ .

**Proof** First, the identity of the associative  $\mathcal{Z}_2$ -algebra  $A$  is clearly a right unit by (9).

Next, for  $x, y, z \in A$ , we have

$$\begin{aligned} 4x \bullet (y \bullet z) &= 2x \bullet (yz_0 + z_0y) = x(yz_0 + z_0y)_0 + (yz_0 + z_0y)_0x \\ &= x(y_0z_0 + z_0y_0) + (y_0z_0 + z_0y_0)x = x(z_0y_0 + y_0z_0)_0 + (z_0y_0 + y_0z_0)_0x \\ &= 2x(z \bullet y)_0 + 2(z \bullet y)_0x = 4x \bullet (z \bullet y), \end{aligned}$$

which proves that the bullet product  $\bullet$  defined by (9) is right commutative.

Thirdly, we have

$$\begin{aligned} 4(y \bullet x) \bullet (x \bullet x) &= 2(yx_0 + x_0y) \bullet (x \bullet x) \\ &= (yx_0 + x_0y)(x \bullet x)_0 + (x \bullet x)_0(yx_0 + x_0y) \\ &= (yx_0 + x_0y)x_0^2 + x_0^2(yx_0 + x_0y) \\ &= (yx_0^2)x_0 + x_0(yx_0^2) + (x_0^2y)x_0 + x_0(x_0^2y) \\ &= (yx_0^2 + x_0^2y)x_0 + x_0(yx_0^2 + x_0^2y) \\ &= (y(x \bullet x)_0 + (x \bullet x)_0y)x_0 + x_0(y(x \bullet x)_0 + (x \bullet x)_0y) \\ &= 2(y \bullet (x \bullet x))x_0 + 2x_0(y \bullet (x \bullet x)) = (y \bullet (x \bullet x)) \bullet x, \end{aligned}$$

which proves that the bullet product  $\bullet$  defined by (9) satisfies the Jordan identity.

Finally, we have

$$\begin{aligned} &4x \bullet (y \bullet (x \bullet x)) - 4(x \bullet y) \bullet (x \bullet x) \\ &= 2x \bullet (y(x \bullet x)_0 + (x \bullet x)_0y) - 2(x \bullet y)(x \bullet x)_0 - 2(x \bullet x)_0(x \bullet y) \\ &= 2x \bullet (yx_0^2 + x_0^2y) - 2(x \bullet y)x_0^2 - 2x_0^2(x \bullet y) \\ &= x(yx_0^2 + x_0^2y)_0 + (yx_0^2 + x_0^2y)_0x - (xy_0 + y_0x)x_0^2 - x_0^2(xy_0 + y_0x) \\ &= \underbrace{xy_0x_0^2 + x_0^2y_0 + y_0x_0^2x}_{1} + \underbrace{x_0^2y_0x - xy_0x_0^2 - y_0xx_0^2}_{2} - \underbrace{x_0^2xy_0 - x_0^2y_0x}_{1} \\ &= xx_0^2y_0 + y_0x_0^2x - y_0xx_0^2 - x_0^2xy_0 \end{aligned} \quad (10)$$

and

$$\begin{aligned}
& 8(x \bullet x) \bullet (y \bullet x) - 8((x \bullet x) \bullet y) \bullet x \\
= & 2(xx_0 + x_0x) \bullet (yx_0 + x_0y) - 4((xx_0 + x_0x) \bullet y) \bullet x \\
= & (xx_0 + x_0x)(yx_0 + x_0y)_0 + (yx_0 + x_0y)_0(xx_0 + x_0x) + \\
& -2((xx_0 + x_0x)y_0 + y_0(xx_0 + x_0x)) \bullet x \\
= & (xx_0 + x_0x)(y_0x_0 + x_0y_0) + (y_0x_0 + x_0y_0)(xx_0 + x_0x) + \\
& -(xx_0y_0 + x_0xy_0 + y_0xx_0 + y_0x_0x)x_0 + \\
& -x_0(xx_0y_0 + x_0xy_0 + y_0xx_0 + y_0x_0x) \\
= & \underbrace{xx_0y_0x_0}_{1} + \underbrace{xx_0x_0y_0}_{2} + \underbrace{x_0xy_0x_0}_{3} + \underbrace{x_0xx_0y_0}_{4} + \underbrace{y_0x_0xx_0}_{5} + \underbrace{y_0x_0x_0x}_{6} + \\
& + \underbrace{x_0y_0xx_0}_{7} + \underbrace{x_0y_0x_0x}_{8} - \underbrace{xx_0y_0x_0}_{9} - \underbrace{x_0xy_0x_0}_{10} - \underbrace{y_0xx_0x_0}_{11} + \\
& - \underbrace{y_0x_0xx_0}_{12} - \underbrace{x_0xx_0y_0}_{13} - \underbrace{x_0x_0xy_0}_{14} - \underbrace{x_0y_0xx_0}_{15} - \underbrace{x_0y_0x_0x}_{16} \\
= & xx_0^2y_0 + y_0x_0^2x - y_0xx_0^2 - x_0^2xy_0
\end{aligned} \tag{11}$$

It follows from (10) and (11) that the the bullet product  $\bullet$  defined by (9) satisfies the Hu-Liu bullet identity.

This completes the proof of Proposition 1.1.  $\square$

**Remark** Except associative  $\mathcal{Z}_2$ -algebras, there exists another class of associative algebras which can be also used to generalize the well-known passage from an associative algebra to a Jordan algebra. The class of associative algebras are called invariant algebras in [3]. In fact, if we replace the sign  $-$  appearing among the four Hu-Liu angle brackets introduced in Section 1.3 of [3] by the sign  $+$ , then resulting new binary operations give a few passages from an invariant algebra to a generalized Jordan algebra.

For convenience, we will use  $A^\dagger$  to denote the right unital generalized Jordan algebra obtained from an associative  $\mathcal{Z}_2$ -algebra  $A = A_0 \oplus A_1$  by using the bullet product (9). Note that  $A_0$  is a ordinary Jordan algebra with respect to the bullet product (9).

**Definition 1.3** Let  $I$  be a subspace of a generalized Jordan algebra  $(J, +, \bullet)$ .

- (i)  $I$  is called an **ideal** of  $J$  if  $I \bullet J + J \bullet I \subseteq I$ .
- (ii)  $I$  is called a **(generalized Jordan) subalgebra** of  $J$  if  $I \bullet I \subseteq I$ .

The **annihilator**  $J^{ann}$  of a generalized Jordan algebra  $(J, +, \bullet)$  over a vector space  $\mathbf{k}$  is defined by

$$J^{ann} := \sum_{x, y \in J} \mathbf{k}(x \bullet y - y \bullet x). \tag{12}$$

For  $x, y, z \in J$ , we have

$$z \bullet (x \bullet y - y \bullet x) = z \bullet (x \bullet y) - z \bullet (y \bullet x) = 0$$

and

$$(x \bullet y - y \bullet x) \bullet z = \left( (x \bullet y) \bullet z - z \bullet (x \bullet y) \right) + \left( z \bullet (y \bullet x) - (y \bullet x) \bullet z \right)$$

by the right commutative law. Hence, the annihilator  $J^{ann}$  is an ideal of  $J$ .

**Definition 1.4** *A generalized Jordan algebra  $J$  is called a **simple generalized Jordan algebra** if  $J \neq J^{ann}$ ,  $J^{ann} \neq 0$  and  $J$  has no ideals which are not equal to  $\{0\}$ ,  $J^{ann}$  and  $J$ .*

If  $(J, +, \bullet)$  is a right unital generalized Jordan algebra, then

$$J^{ann} = \{ a \mid 1 \bullet a = 0 \}$$

and

$$\{ 1 + a \mid a \in J^{ann} \} = \text{the set of all right units of } J,$$

where 1 is a right unit of  $J$ .

Let  $(J, +, \bullet)$  be a generalized Jordan algebra. The **long associator**  $[x, y, z]_\ell$  is defined by

$$[x, y, z]_\ell := x \bullet (y \bullet z) - (x \bullet y) \bullet z - 2z \bullet (y \bullet x) + 2(z \bullet y) \bullet x, \quad (13)$$

where  $x, y, z \in J$ . By the Hu-Liu identity, we have

$$[x, y, x \bullet x]_\ell = 0 \quad \text{for } x, y \in J. \quad (14)$$

After linearizing (14), we get

$$[x, y, x \bullet z]_\ell + [x, y, z \bullet x]_\ell + [z, y, x \bullet x]_\ell = 0 \quad (15)$$

and

$$[x, y, w \bullet z + z \bullet w]_\ell + [w, y, z \bullet x + x \bullet z]_\ell + [z, y, x \bullet w + w \bullet x]_\ell = 0, \quad (16)$$

where  $x, y, z, w \in J$ .

For  $a \in J$ , let  $L_a : J \rightarrow J$  and  $R_a : J \rightarrow J$  be the **left multiplication** and the **right multiplication**; that is

$$L_a(x) := a \bullet x, \quad R_a(x) := x \bullet a \quad \text{for all } x \in J.$$

For convenience, we also define  $S_a$  by

$$S_a := L_a + R_a \quad \text{for all } x \in J.$$

It follows from (5), (6) and (7) that left multiplications and right multiplications have the following properties:

$$L_x L_y = L_x R_y, \quad R_{x \bullet y} = R_{y \bullet x} \quad \text{for all } x, y \in J, \quad (17)$$

$$R_x R_{x \bullet x} = R_{x \bullet x} R_x \quad \text{for all } x \in J \quad (18)$$

and

$$L_x R_{x \bullet x} - R_{x \bullet x} L_x = 2L_{x \bullet x} R_x - 2R_x L_{x \bullet x} \quad \text{for all } x \in J. \quad (19)$$

The long associator  $[x, y, z]_\ell$  can also be written as

$$[x, y, z]_\ell \begin{cases} = (R_{y \bullet z} - R_z R_y - 2L_z L_y + 2L_{z \bullet y})(x), \\ = (L_x R_z - R_z L_x - 2L_z R_x + 2R_x L_z)(y), \\ = (L_x L_y - L_{x \bullet y} - 2R_{y \bullet x} + 2R_x R_y)(z). \end{cases} \quad (20)$$

By (20), (16) is equivalent to

$$\begin{aligned} & (L_x L_y - L_{x \bullet y} - 2R_{y \bullet x} + 2R_x R_y)S_z + \\ & + 2(R_{y \bullet (z \bullet x)} - R_{z \bullet x} R_y - L_{z \bullet x + x \bullet z} L_y + L_{(z \bullet x + x \bullet z) \bullet y}) + \\ & + (L_z L_y - L_{z \bullet y} - 2R_{y \bullet z} + 2R_z R_y)S_x = 0. \end{aligned} \quad (21)$$

Letting  $x = z$  in (21), we get

$$\begin{aligned} & (L_x L_y - L_{x \bullet y} - 2R_{y \bullet x} + 2R_x R_y)S_x + \\ & + R_{(x \bullet x) \bullet y} - R_{x \bullet x} R_y - 2L_{x \bullet x} L_y + 2L_{(x \bullet x) \bullet y} = 0. \end{aligned} \quad (22)$$

Recall from Section 9 of Chapter II in [1] that a vector space  $V$  over a Jordan algebra  $(\mathcal{J}, +, \odot)$  is called a **Jordan bimodule** if there is a bilinear map  $(v, a) \mapsto va$  from  $V \times \mathcal{J} \rightarrow V$  satisfying

$$(v(a \odot a))a = (va)(a \odot a) \quad (23)$$

and

$$2((va)b)a + v((a \odot a) \odot b) = 2(va)(a \odot b) + (vb)(a \odot a) \quad (24)$$

for  $v \in V$  and  $a, b \in \mathcal{J}$ .

The next proposition shows that a generalized Jordan algebra structure is obtained by combining of a Jordan algebra and a Jordan bimodule.

**Proposition 1.2** *If  $(J, +, \bullet)$  is a generalized Jordan algebra over a field  $\mathbf{k}$ , then the annihilator  $J^{ann}$  becomes a bimodule over the Jordan algebra  $\frac{J}{J^{ann}}$  under the following bimodule action:*

$$u\bar{x} := u \bullet x \quad \text{for } u \in J^{ann}, x \in J \text{ and } \bar{x} := x + J^{ann} \in \frac{J}{J^{ann}}. \quad (25)$$

**Proof** Note that  $\left(\frac{J}{J^{ann}}, +, \odot\right)$  is a Jordan algebra, where the product  $\odot$  is defined by

$$\bar{x} \odot \bar{y} := \overline{x \bullet y} \quad \text{for } x, y \in J.$$

By the right commutative property, the action (25) is well-defined. According to (23) and (24), we need to prove

$$(u(\bar{x} \odot \bar{x}))\bar{x} = (u\bar{x})(\bar{x} \odot \bar{x}) \quad (26)$$

and

$$2((u\bar{x})\bar{y})\bar{x} + u((\bar{x} \odot \bar{x}) \odot \bar{y}) = 2(u\bar{x})(\bar{x} \odot \bar{y}) + (u\bar{y})(\bar{x} \odot \bar{x}) \quad (27)$$

for  $u \in J^{ann}$  and  $x, y \in J$ .

Since

$$u(\bar{x} \odot \bar{x}) = u \overline{x \bullet x} = u \bullet (x \bullet x) = (R_{x \bullet x}|J^{ann})(u),$$

(26) follows from (18).

Using  $L_x|J^{ann} = 0$  and  $S_x|J^{ann} = R_x|J^{ann}$ , we get from (22) that

$$-2R_{y \bullet x}R_x(u) + 2R_xR_yR_x(u) = R_{(x \bullet x) \bullet y}(u) - R_{x \bullet x}R_y(u)$$

or

$$-2(u\bar{x})(\bar{x} \odot \bar{y}) + 2((u\bar{x})\bar{y})\bar{x} = u((\bar{x} \odot \bar{x}) \odot \bar{y}) - (u\bar{y})(\bar{x} \odot \bar{x}),$$

which is (27). □

We now give a way of constructing a generalized Jordan algebra structure from a Jordan algebra and a Jordan bimodule over the Jordan algebra.

**Proposition 1.3** *If  $(\mathcal{J}, +, \odot)$  is a Jordan algebra and  $V$  a Jordan bimodule over the Jordan algebra  $(\mathcal{J}, +, \odot)$ , then the vector space direct sum*

$$J := \mathcal{J} \oplus V = \{ (a, v) \mid a \in \mathcal{J} \text{ and } v \in V \}$$

*becomes a generalized Jordan algebra under the following bullet product:*

$$(a, v) \bullet (b, u) := (a \odot b, vb) \quad a, b \in \mathcal{J} \text{ and } v, u \in V. \quad (28)$$

**Proof** Let  $(a, v), (b, u), (c, w) \in \mathcal{J} \oplus V$ , where  $a, b, c \in \mathcal{J}$  and  $v, u, w \in V$ .

First, we have

$$\begin{aligned} (a, v) \odot ((b, u) \odot (c, w)) &= (a, v) \odot (b \odot c, uc) = (a \odot (b \odot c), v(b \odot c)) \\ &= (a \odot (c \odot b), v(c \odot b)) = (a, v) \odot (c \odot b, wb) = (a, v) \odot ((c, w) \odot (b, u)), \end{aligned}$$

which proves that the bullet product  $\bullet$  defined by (28) is right commutative.

Next, we have

$$\begin{aligned} & \left( (b, u) \bullet (a, v) \right) \bullet \left( (a, v) \bullet (a, v) \right) = (b \odot a, ua) \bullet (a \odot a, va) \\ & = \left( (b \odot a) \odot (a \odot a), (ua)(a \odot a) \right) \end{aligned} \quad (29)$$

and

$$\begin{aligned} & \left( (b, u) \bullet ((a, v) \bullet (a, v)) \right) \bullet (a, v) = \left( (b, u) \bullet (a \odot a, va) \right) \bullet (a, v) \\ & = \left( b \odot (a \odot a), u(a \odot a) \right) \bullet (a, v) \\ & = \left( (b \odot (a \odot a)) \odot a, (u(a \odot a))a \right). \end{aligned} \quad (30)$$

By (29) and (30), the bullet product  $\bullet$  defined by (28) satisfies the Jordan identity.

Finally, we have

$$\begin{aligned} & (a, v) \bullet \left( (b, u) \bullet ((a, v) \bullet (a, v)) \right) - \left( (a, v) \bullet (b, u) \right) \bullet \left( (a, v) \bullet (a, v) \right) \\ & = (a, v) \bullet \left( (b, u) \bullet (a \odot a, va) \right) - \left( (a \odot b, vb) \bullet (a \odot a, va) \right) \\ & = (a, v) \bullet \left( b \odot (a \odot a), u(a \odot a) \right) - \left( (a \odot b) \odot (a \odot a), (vb)(a \odot a) \right) \\ & = \left( a \odot (b \odot (a \odot a)), v(b \odot (a \odot a)) \right) - \left( (a \odot b) \odot (a \odot a), (vb)(a \odot a) \right) \\ & = \left( 0, v((a \odot a) \odot b) - (vb)(a \odot a) \right) \end{aligned} \quad (31)$$

and

$$\begin{aligned} & 2 \left( ((a, v) \bullet (a, v)) \bullet ((b, u) \bullet (a, v)) \right) - 2 \left( ((a, v) \bullet (a, v)) \bullet (b, u) \right) \bullet (a, v) \\ & = 2 \left( (a \odot a, va) \bullet (b \odot a, ua) \right) - 2 \left( (a \odot a, va) \bullet (b, u) \right) \bullet (a, v) \\ & = 2 \left( (a \odot a) \odot (b \odot a), (va)(b \odot a) \right) - 2 \left( (a \odot a) \odot b, (va)b \right) \bullet (a, v) \\ & = 2 \left( (a \odot a) \odot (b \odot a), (va)(b \odot a) \right) - 2 \left( ((a \odot a) \odot b) \odot a, ((va)b)a \right) \\ & = \left( 0, 2(va)(a \odot b) \right) - 2((va)b)a. \end{aligned} \quad (32)$$

By (31) and (32), the bullet product  $\bullet$  defined by (28) satisfies the Hu-Liu identity.

This completes the proof of Proposition 1.3.

□



## 2 Free Associative $\mathcal{Z}_2$ -Algebra

Let  $\check{X}$  and  $\check{\Theta}$  be two disjoint sets. Let  $\mathcal{FA}[\check{X} \cup \check{\Theta}]$  denote the free unital  $\mathbf{k}$ -associative algebra on the set  $\check{X} \cup \check{\Theta}$ ; that is,  $\mathcal{FA}[\check{X} \cup \check{\Theta}]$  is a vector space over  $\mathbf{k}$  with a basis consisting of all **monomials**  $\check{u}_1 \cdots \check{u}_n$  for all  $n \geq 0$  ( the **empty product** for  $n = 0$  serving as unit 1) and all  $\check{u}_i \in \check{X} \cup \check{\Theta}$ , with the associative product determined by linearity and juxtaposition:

$$(\check{u}_1 \cdots \check{u}_n)(\check{u}_{n+1} \cdots \check{u}_{n+m}) = \check{u}_1 \cdots \check{u}_n \check{u}_{n+1} \cdots \check{u}_{n+m}.$$

The  $\Theta$ -degree  $\deg_{\Theta}(\check{u}_1 \cdots \check{u}_n)$  of a monomial  $\check{u}_1 \cdots \check{u}_n$  is defined by

$$\deg_{\Theta}(\check{u}_1 \cdots \check{u}_n) := \begin{cases} 0 & \text{if } n = 0, \\ |\{\check{u}_i \mid \check{u}_i \in \Theta \text{ and } 1 \leq i \leq n\}| & \text{if } n \geq 1. \end{cases}$$

Let

$$I_{\Theta} := \sum_{\substack{\check{u} \text{ is a monomial} \\ \text{and } \deg_{\Theta}(\check{u}) \geq 2}} \mathbf{k}\check{u}.$$

Then  $I_{\Theta}$  is an ideal of the free associative algebra  $\mathcal{FA}[\check{X} \cup \check{\Theta}]$ . We use  $\mathcal{FA}_2[X \cup \Theta]$  to denote the quotient associative algebra  $\frac{\mathcal{FA}[\check{X} \cup \check{\Theta}]}{I_{\Theta}}$ , where

$$X := \{x \mid x = \check{x} + I_{\Theta} \text{ and } \check{x} \in \check{X}\}$$

and

$$\Theta := \{\theta \mid \theta = \check{\theta} + I_{\Theta} \text{ and } \check{\theta} \in \check{\Theta}\}.$$

Then  $\mathcal{FA}_2[X \cup \Theta]$  is an associative  $\mathcal{Z}_2$ -algebra whose even part  $\mathcal{FA}_2[X \cup \Theta]_0$  and odd part  $\mathcal{FA}_2[X \cup \Theta]_1$  are given by

$$\mathcal{FA}_2[X \cup \Theta]_0 := \bigoplus_{\substack{n \geq 0 \\ x_{i_1}, \dots, x_{i_n} \in X}} \mathbf{k}x_{i_1} \cdots x_{i_n}$$

and

$$\mathcal{FA}_2[X \cup \Theta]_1 := \bigoplus_{\substack{n, m \geq 0 \\ x_{i_1}, \dots, x_{i_n} \in X \\ x_{j_1}, \dots, x_{j_m} \in X \\ \theta_{x_{i_1}, \dots, x_{i_n}} \in \Theta}} \mathbf{k}x_{i_1} \cdots x_{i_n} \theta_{x_{i_1}, \dots, x_{i_n}} x_{j_1} \cdots x_{j_m}.$$

$\mathcal{FA}_2[X \cup \Theta]$  is called the **free associative  $\mathcal{Z}_2$ -algebra** on the set  $X \cup \Theta$ . The next proposition gives the universal property of  $\mathcal{FA}_2[X \cup \Theta]$ .

**Proposition 2.1** *If  $\phi$  is a map from the set  $X \cup \Theta$  to an associative  $\mathcal{Z}_2$ -algebra  $A = A_0 \oplus A_1$  such that  $\phi(\Theta) \subseteq A_1$ , then  $\phi$  can be extended uniquely to an associative algebra homomorphism from  $\mathcal{FA}_2[X \cup \Theta]$  to  $A$ ; that is, there exists a unique associative algebra homomorphism  $\check{\phi} : \mathcal{FA}_2[X \cup \Theta] \rightarrow A$  such that  $\check{\phi}|(X \cup \Theta) = \phi$ .*

**Proof**  $\check{\phi}$  is clearly unique. We need only to prove the existence of  $\check{\phi}$ . By the universal property of the free unital associative algebra  $\mathcal{FA}[\check{X} \cup \check{\Theta}]$ , there exists an associative algebra homomorphism  $\check{\phi} : \mathcal{FA}[\check{X} \cup \check{\Theta}] \rightarrow A$  such that

$$\check{\phi}(\check{y}) = \phi(y), \quad (33)$$

where  $y = \check{y} + \check{\Theta}$  and  $\check{y} \in \check{X} \cup \check{\Theta}$ . If  $\check{u}$  is a monomial with  $\deg_{\Theta}(\check{u}) \geq 2$ , then there exist  $\check{y}_{i_1}, \dots, \check{y}_{i_t} \in \check{X} \cup \check{\Theta}$  and  $\check{\theta}_1, \check{\theta}_2 \in \check{\Theta}$  such that

$$\check{u} = \check{y}_{i_1} \cdots \check{y}_{i_n} \check{\theta}_1 \check{y}_{i_{n+1}} \cdots \check{y}_{i_{n+m}} \check{\theta}_2 \check{y}_{i_{n+m+1}} \cdots \check{y}_{i_t}.$$

Since  $\phi(\theta_i) \in A_1$  for  $i = 1$  and  $2$ , it follows from (33) that

$$\begin{aligned} \check{\phi}(\check{u}) &= \check{\phi}(\check{y}_{i_1}) \cdots \check{\phi}(\check{y}_{i_n}) \check{\phi}(\check{\theta}_1) \check{\phi}(\check{y}_{i_{n+1}}) \cdots \check{\phi}(\check{y}_{i_{n+m}}) \check{\phi}(\check{\theta}_2) \check{\phi}(\check{y}_{i_{n+m+1}}) \cdots \check{\phi}(\check{y}_{i_t}) \\ &= y_{i_1} \cdots y_{i_n} \theta_1 y_{i_{n+1}} \cdots y_{i_{n+m}} \theta_2 y_{i_{n+m+1}} \cdots y_{i_t} = 0, \end{aligned}$$

which proves that  $I_{\Theta} \subseteq \text{Ker} \check{\phi}$ . Thus,  $\check{\phi}$  induces an associative algebra homomorphism

$$\check{\phi} : \mathcal{FA}_2[X \cup \Theta] = \frac{\mathcal{FA}[\check{X} \cup \check{\Theta}]}{I_{\Theta}} \rightarrow A$$

such that  $\check{\phi}(y) = \check{\phi}(\check{y}) = \phi(y)$  for  $y \in X \cup \Theta$ . This proves Proposition 2.1.  $\square$

### 3 Generalized Cohn's Theorem

Let  $(\mathcal{FA}_2[X \cup \Theta]^{\dagger}, +, \bullet)$  be the right unital generalized Jordan algebra obtained from the free associative  $\mathcal{Z}_2$ -algebra  $\mathcal{FA}_2[X \cup \Theta]$ , where the bullet product  $\bullet$  is defined by (9). The free associative  $\mathcal{Z}_2$ -algebra  $\mathcal{FA}_2[X \cup \Theta]$  has a unique **reversal involution**  $*$  such that

$$1^* := 1, \quad (y_1 y_2 \cdots y_n)^* := y_n^* \cdots y_2^* y_1^* \quad \text{for } y_1, \dots, y_n \in X \cup \Theta.$$

It is clear that the reversal involution  $*$  preserves the  $\mathcal{Z}_2$ -grading of  $\mathcal{FA}_2[X \cup \Theta]$ . An element  $a$  of  $\mathcal{FA}_2[X \cup \Theta]$  is said to be **reversible** if  $a^* = a$ . If  $a$  and  $b$  are two reversible elements of  $\mathcal{FA}_2[X \cup \Theta]$ , then

$$\begin{aligned} (a \bullet b)^* &= \left( \frac{1}{2}(ab_0 + b_0a) \right)^* = \frac{1}{2} \left( (b_0)^* a^* + a^* (b_0)^* \right) \\ &= \frac{1}{2} \left( (b^*)_0 a^* + a^* (b^*)_0 \right) = \frac{1}{2} (b_0a + ab_0) = a \bullet b. \end{aligned}$$

This proves that the reversible elements of  $\mathcal{FA}_2[X \cup \Theta]$  form a right unital generalized Jordan subalgebra  $\mathcal{H}(\mathcal{FA}_2[X \cup \Theta], *)$  of  $\mathcal{FA}_2[X \cup \Theta]^\dagger$ .

In the remaining of this section, we fix an ordering of  $X \cup \Theta$  with the following property:

$$\theta < x \quad \text{for } \theta \in \Theta \text{ and } x \in X.$$

For  $y_1, \dots, y_n \in X \cup \Theta$ , let

$$\{y_1, y_2, \dots, y_n\} := \frac{1}{2}(y_1 y_2 \cdots y_n + y_n \cdots y_2 y_1).$$

If  $x_1, x_2, x_3, x_4$  are distinct elements of  $X$  and  $\theta \in \Theta$ , then  $\{x_1, x_2, x_3, x_4\}$  with  $x_1 < x_2 < x_3 < x_4$  is called an **even tetrad**, and  $\{\theta, x_1, x_2, x_3\}$  with  $x_1 < x_2 < x_3$  is called an **odd tetrad**.

The following proposition gives a generalization of Cohn's Theorem.

**Proposition 3.1** *The right unital generalized Jordan algebra  $\mathcal{H}(\mathcal{FA}_2[X \cup \Theta], *)$  of reversible elements of the free associative  $\mathbb{Z}_2$ -algebra  $\mathcal{FA}_2[X \cup \Theta]$  coincides with the generalized Jordan subalgebra  $\mathcal{H}'$  of  $\mathcal{FA}_2[X \cup \Theta]^\dagger$  generated by 1,  $X \cup \Theta$  and all even tetrads and odd tetrads.*

**Proof** It is clear that  $\mathcal{H}' \subseteq \mathcal{H}(\mathcal{FA}_2[X \cup \Theta], *)$ . Hence, we need only to prove

$$\mathcal{H}(\mathcal{FA}_2[X \cup \Theta], *) \subseteq \mathcal{H}'. \quad (34)$$

It is easy to check that  $\mathcal{H}(\mathcal{FA}_2[X \cup \Theta], *)$  is spanned by the set

$$\left\{ \{x_1, x_2, \dots, x_n\}, \{x_1, x_2, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} \mid \begin{array}{l} n, m \geq 0, \theta \in \Theta \\ x_1, \dots, x_m \in X \end{array} \right\},$$

where

$$\{x_1, x_2, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} := \begin{cases} \{x_1, x_2, \dots, x_n, \theta\} & \text{if } n = m, \\ \{\theta, x_1, x_2, \dots, x_m\} & \text{if } n = 0. \end{cases}$$

Hence, in order to prove (34), it is enough to prove

$$\{x_1, x_2, \dots, x_n\} \equiv 0 \pmod{\mathcal{H}'} \quad \text{for } n \geq 0 \quad (35)$$

and

$$\{x_1, x_2, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} \equiv 0 \pmod{\mathcal{H}'} \quad \text{for } 0 \leq n \leq m. \quad (36)$$

It follows from Chon's Theorem that (35) holds. Thus, the only thing we need to prove is (36). We will prove (36) by induction on  $m$ .

Clearly, we have

$$\{\theta\} = \theta, \quad \{x_1, \theta\} = \{\theta, x_1\} = \theta \bullet x_1, \quad (37)$$

$$\{\theta, x_1, x_2\} = \{x_2, x_1, \theta\} = (\theta \bullet x_1) \bullet x_2 - (\theta \bullet x_2) \bullet x_1 + \theta \bullet (x_1 \bullet x_2) \quad (38)$$

and

$$\{x_1, \theta, x_2\} = (\theta \bullet x_1) \bullet x_2 + (\theta \bullet x_2) \bullet x_1 - \theta \bullet (x_1 \bullet x_2). \quad (39)$$

It follows from (37), (38) and (39) that (36) holds for  $m = 0, 1$  and  $2$ .

We now assume that  $m \geq 3$  and

$$\{z_1, \dots, z_s, \theta, z_{s+1}, \dots, z_t\} \equiv 0 \pmod{\mathcal{H}'}, \quad (40)$$

where  $0 \leq s \leq t < m$ ,  $\theta \in \Theta$  and  $z_1, \dots, z_t \in X$ . If  $0 \leq h < m - n$  and  $m - n \geq 1$ , then (35) and (40) imply

$$\begin{aligned} \mathcal{H}' &\ni 8\{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_{n+h}\} \bullet \{x_{n+h+1}, \dots, x_m\} \\ &= 2(x_1 \cdots x_n \theta x_{n+1} \cdots x_{n+h} + x_{n+h} \cdots x_{n+1} \theta x_n \cdots x_1) \bullet \\ &\quad \bullet (x_{n+h+1} \cdots x_m + x_m \cdots x_{n+h+1}) \\ &= (x_1 \cdots x_n \theta x_{n+1} \cdots x_{n+h} + x_{n+h} \cdots x_{n+1} \theta x_n \cdots x_1) \cdot \\ &\quad \cdot (x_{n+h+1} \cdots x_m + x_m \cdots x_{n+h+1}) + \\ &\quad + (x_{n+h+1} \cdots x_m + x_m \cdots x_{n+h+1}) \cdot \\ &\quad \cdot (x_1 \cdots x_n \theta x_{n+1} \cdots x_{n+h} + x_{n+h} \cdots x_{n+1} \theta x_n \cdots x_1) \\ &= \underbrace{x_1 \cdots x_n \theta x_{n+1} \cdots x_m}_1 + \underbrace{x_1 \cdots x_n \theta x_{n+1} \cdots x_{n+h} x_m \cdots x_{n+h+1}}_2 + \\ &\quad + \underbrace{x_{n+h} \cdots x_{n+1} \theta x_n \cdots x_1 x_{n+h+1} \cdots x_m}_3 + \\ &\quad + \underbrace{x_{n+h} \cdots x_{n+1} \theta x_n \cdots x_1 x_m \cdots x_{n+h+1}}_4 + \\ &\quad + \underbrace{x_{n+h+1} \cdots x_m x_1 \cdots x_n \theta x_{n+1} \cdots x_{n+h}}_4 + \\ &\quad + \underbrace{x_{n+h+1} \cdots x_m x_{n+h} \cdots x_{n+1} \theta x_n \cdots x_1}_2 + \\ &\quad + \underbrace{x_m \cdots x_{n+h+1} x_1 \cdots x_n \theta x_{n+1} \cdots x_{n+h}}_3 + \underbrace{x_m \cdots x_{n+1} \theta x_n \cdots x_1}_1 \\ &= 2\{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} + \\ &\quad + 2\{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_{n+h}, x_m, \dots, x_{n+h+1}\} + \\ &\quad + 2\{x_{n+h}, \dots, x_{n+1}, \theta, x_n, \dots, x_1, x_{n+h+1}, \dots, x_m\} + \\ &\quad + 2\{x_{n+h}, \dots, x_{n+1}, \theta, x_n, \dots, x_1, x_m, \dots, x_{n+h+1}\} \end{aligned}$$

or

$$\begin{aligned} &\{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} + \\ &\quad + \{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_{n+h}, x_m, \dots, x_{n+h+1}\} + \\ &\quad + \{x_{n+h}, \dots, x_{n+1}, \theta, x_n, \dots, x_1, x_{n+h+1}, \dots, x_m\} + \\ &\quad + \{x_{n+h}, \dots, x_{n+1}, \theta, x_n, \dots, x_1, x_m, \dots, x_{n+h+1}\} \\ &\equiv 0 \pmod{\mathcal{H}'} \quad \text{for } 0 \leq h < m - n \text{ and } m - n \geq 1. \end{aligned} \quad (41)$$

Let  $h = n = 0$  in (41), we get

$$\begin{aligned} & \{\theta, x_1, \dots, x_m\} + \{\theta, x_m, \dots, x_1\} + \\ & + \{\theta, x_1, \dots, x_m\} + \{\theta, x_m, \dots, x_1\} \equiv 0 \pmod{\mathcal{H}'} \end{aligned}$$

or

$$\{\theta, x_1, x_2, \dots, x_m\} \equiv -\{\theta, x_m, \dots, x_2, x_1\} \pmod{\mathcal{H}'}. \quad (42)$$

Let  $h = m - n - 1$  in (41), we get

$$\begin{aligned} & \{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} + \{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_{m-1}, x_m\} + \\ & + \{x_{m-1}, \dots, x_{n+1}, \theta, x_n, \dots, x_1, x_m\} + \\ & + \{x_{m-1}, \dots, x_{n+1}, \theta, x_n, \dots, x_1, x_m\} \equiv 0 \pmod{\mathcal{H}'} \end{aligned}$$

or

$$\begin{aligned} & \{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} \equiv -\{x_{m-1}, \dots, x_{n+1}, \theta, x_n, \dots, x_1, x_m\} \\ \equiv & -\{x_m, x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_{m-1}\} \pmod{\mathcal{H}'}. \end{aligned} \quad (43)$$

It follows that

$$\begin{aligned} & \{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} \stackrel{(43)}{\equiv} -\{x_m, x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_{m-1}\} \\ \stackrel{(43)}{\equiv} & (-1)^2 \{x_{m-1}, x_m, x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_{m-2}\} \stackrel{(43)}{\equiv} \dots \stackrel{(43)}{\equiv} \\ \stackrel{(43)}{\equiv} & (-1)^{m-n} \{x_{n+1}, \dots, x_m, x_1, \dots, x_n, \theta\} \pmod{\mathcal{H}'} \quad (44) \\ \equiv & (-1)^{m-n} \{\theta, x_n, \dots, x_1, x_m, \dots, x_{n+1}\} \pmod{\mathcal{H}'} \quad (45) \\ \stackrel{(42)}{\equiv} & (-1)^{m-n+1} \{\theta, x_{n+1}, \dots, x_m, x_1 \dots, x_n\} \\ \stackrel{(43)}{\equiv} & (-1)^{m-n+2} \{x_n, \theta, x_{n+1}, \dots, x_m, x_1 \dots, x_{n-1}\} \stackrel{(43)}{\equiv} \dots \stackrel{(43)}{\equiv} \\ \stackrel{(43)}{\equiv} & (-1)^{m-n+(n+1)} \{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} \\ = & (-1)^{m+1} \{x_1, \dots, x_n, \theta, x_{n+1}, \dots, x_m\} \pmod{\mathcal{H}'}. \end{aligned} \quad (46)$$

By (46), (36) holds for even  $m$ . From now on assume that  $m$  is odd and  $m \geq 3$ . By (45), to complete the proof of Proposition 3.1, the only remaining thing we need to prove is

$$\{\theta, x_1, \dots, x_m\} \equiv 0 \pmod{\mathcal{H}'} \quad \text{for } m \text{ is odd and } m \geq 3. \quad (47)$$

Letting  $n = 0$  and  $h = m - 2$  in (41), we get

$$\begin{aligned} & \{\theta, x_1, \dots, x_m\} + \{\theta, x_1, \dots, x_{m-2}, x_m, x_{m-1}\} + \\ & + \{x_{m-2}, \dots, x_1, \theta, x_{m-1}, x_m\} + \\ & + \{x_{m-2}, \dots, x_1, \theta, x_m, x_{m-1}\} \equiv 0 \pmod{\mathcal{H}'} \end{aligned} \quad (48)$$

It follows from (44) and (48) that

$$\{\theta, x_1, \dots, x_m\} \equiv -\{\theta, x_1, \dots, x_{m-2}, x_m, x_{m-1}\} \pmod{\mathcal{H}'}. \quad (49)$$

By (42), we have

$$\{\theta, x_1, \dots, x_m\} \equiv -\{x_1, \dots, x_{m-1}, x_m, \theta\} \pmod{\mathcal{H}'}. \quad (50)$$

Since the cycle  $\begin{pmatrix} \theta & x_1 & x_2 & \cdots & x_{m-1} & x_m \\ x_1 & x_2 & x_3 & \cdots & x_m & \theta \end{pmatrix}$  and the transposition  $\begin{pmatrix} \theta & x_1 & \cdots & x_{m-2} & x_{m-1} & x_m \\ \theta & x_1 & \cdots & x_{m-2} & x_m & x_{m-1} \end{pmatrix}$  generate the symmetric group  $\mathbf{S}_{m+1}$  on the set  $\{\theta, x_1, \dots, x_m\}$ , it follows from (49) and (50) that

$$\{\theta, x_1, \dots, x_m\} \equiv (\text{sign } \sigma) \{\sigma(\theta), \sigma(x_1), \dots, \sigma(x_m)\} \pmod{\mathcal{H}'} \quad (51)$$

for any permutation  $\sigma = \begin{pmatrix} \theta & x_1 & x_2 & \cdots & x_m \\ \sigma(\theta) & \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_m) \end{pmatrix}$  in  $\mathbf{S}_{m+1}$ , where  $\text{sign } \sigma$  denotes the sign of the permutation  $\sigma$ .

If  $m = 3$ , then there is a permutation  $\sigma = \begin{pmatrix} \theta & x_1 & x_2 & x_3 \\ \theta & \sigma(x_1) & \sigma(x_2) & \sigma(x_3) \end{pmatrix}$  on the set  $\{\theta, x_1, x_2, x_3\}$  such that  $\sigma(x_1) < \sigma(x_2) < \sigma(x_3)$ . Using this permutation  $\sigma$  and (51), we get

$$\{\theta, x_1, x_2, x_3\} \equiv \pm \{\theta, \sigma(x_1), \sigma(x_2), \sigma(x_3)\} \equiv 0 \pmod{\mathcal{H}'},$$

which proves that (47) holds for  $m = 3$ .

In the remaining of the proof, we assume that  $m$  is odd and  $m \geq 5$ . Let  $n = 0$  and  $h = 3$  in (41), we get

$$\begin{aligned} & \{\theta, x_1, \dots, x_m\} + \{\theta, x_1, x_2, x_3, x_m, \dots, x_5, x_4\} + \\ & + \{x_3, x_2, x_1, \theta, x_4, x_5, \dots, x_m\} + \\ & + \{x_3, x_2, x_1, \theta, x_m, \dots, x_5, x_4\} \equiv 0 \pmod{\mathcal{H}'}. \end{aligned} \quad (52)$$

Let  $\tau$  and  $\mu$  be two permutations on the set  $\{\theta, x_1, x_2, \dots, x_m\}$  defined by

$$\tau := \begin{pmatrix} \theta & x_1 & x_2 & \cdots & x_{m-1} & x_m \\ x_1 & x_2 & x_3 & \cdots & x_m & \theta \end{pmatrix}$$

and

$$\mu := \begin{pmatrix} x_3 & x_2 & x_1 & \theta & x_m & x_{m-1} & \cdots & x_4 \\ \theta & x_1 & x_2 & x_3 & x_m & x_{m-1} & \cdots & x_4 \end{pmatrix}.$$

Then  $\mu$  is even and

$$\tau^4 = \begin{pmatrix} \theta & x_1 & \cdots & x_{m-4} & x_{m-3} & x_{m-2} & x_{m-1} & x_m \\ x_4 & x_5 & \cdots & x_m & \theta & x_1 & x_2 & x_3 \end{pmatrix}.$$

Using  $\tau$ ,  $\mu$  and (51), (52) implies

$$4\{\theta, x_1, \dots, x_m\} \equiv 0 \pmod{\mathcal{H}'},$$

which proves that (47) holds for odd  $m \geq 5$ .

This completes the proof of Proposition 3.1.

□

## References

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